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Hypermass Generalization of Einstein's Gravitation Theory

JAMES D. EDMONDS, Jr.†

Space Science Division, Ames Research Center, NASA, Moffet Field, California 94035, *and Department of Aeronautics and Astronautics, Stanford University, Stanford, California* 94305

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Abstract

The curvilinear invariant quaternion formalism is examined for curved space time. Einstein's gravitation equation is shown to have a simple and natural form in this notation. The hypermass generalization of particle mass, which was generated in our studies of the Dirac equation, is incorporated in gravitation by generalizing Einstein's equation. Covariance requires that the gravitational constant be generalized to an invariant quaternion when the mass is. The modification appears minor and of no importance cosmologically, unless one begins considering time and mass dependence of G.

1. *Introduction*

Many notations and formalisms have been developed which describe the same physical laws, quantum and classical (Utiyama, 1956; Brown, 1962; Newman and Penrose, 1962; Sachs, 1968; Carmeli, 1972). The quaternion formalism invented by Hamilton in the previous century predates the vector formalism, but has been largely overshadowed by the latter since the beginning of this century (Bork, 1966). The quarternion formalism has been recently reviewed by Rastall (1964). It has continued over the years to attract a small number of devotees [see references in Edmonds (1972)].

Our recent work with this formalism (Edmonds, 1972) has led to a generalization of the Dirac equation, produced by adding hypermass elements to the usual inertial mass. This generalization is not physically well motivated. It is motivated largely by the structure of the Dirac equation in quaternion notation, in which the four components of the wave function seem to be a degenerate state of a system with eight wave function components. This 'degeneracy' is lifted by slightly modifying the Dirac equation by introducing quaternion mass. The resulting generalized Dirac equation (Edmonds, 1973a)

$$ih\,\partial\psi_a = \psi_v mc, \qquad ih\,\partial^+\psi_v = \psi_a m^+ c \tag{1.1}$$

† NASA-ASEE Summer Faculty Fellow, 1972 and 1973. Address 1972–1973, Department of Physics, Oregon State University, Corvallis, Oregon 97331.

represents a spin-$\frac{1}{2}$ particle with a new spin-like quantum number, tumble (Edmonds, 1973b). The four particle rest states become spin-up tumble-up, spin-down tumble-up, spin-up tumble-down, and spin-down tumble-down, with four corresponding states for negative energy (antiparticle) solutions. The invariant hypermass, $m^0 e_0 + m^k e_k$, has only two independent parts m^0 and $m \equiv m^k$ for an isotropic space. We have demonstrated how the usual results of quantum theory look in this formalism by solving the Dirac hydrogen atom (Edmonds, 1973c) using quaternion wave functions.

At present the physical relevance of the hypermass concept is untested. It obviously has consequences for the muon electron and two neutrino problems. The details of these consequences remain to be explored, but hopefully some quantitative results will be forthcoming. We have also considered the formal problem of quaternion observables in quantum theory (Edmonds, 1973d), which seems incompatible with a single particle approximation.

In an earlier paper (Edmonds, 1972), we made a tentative effort to develop a curved space-time quaternion formalism. The approach taken there proved unproductive. Rastall (1962) has reviewed the conventional quaternion formulation of curved space time. In this paper we build on his formulation, using our earlier notation, to write Einstein's gravitation equation in quaternion form. We then show how the hypermass concept, which came out of our quantum studies, can be used to generalize the gravitation equation in a covariant way. This generalization will likely not lead to any startling cosmological consequences since quantum electrodynamics gives the Lamb shift to one part in 10^8. However, at this stage one just does not know. Such things as high flux density hypermass neutrinos could possibly produce cosmic consequences in the evolution of the universe.

It is of some interest to note that in this notation Einstein's gravity equation (as was found for Dirac's quantum equation) takes a simple and natural form, again reinforcing the conviction that the complex quaternions are nature's natural numbers.

2. *Curved Space-Time Quaternions*

In flat space-time, using cartesian coordinates, an event can be represented by $x = x^\mu e_\mu$, $x^0 = ct$, $\{x^k\} = \{x, y, z\}$. For curvilinear coordinates or curved space-time this idea must be generalized. We can do this simply as follows. We introduce the *Invariance Principle*. By definition this states that all physical laws written in quaternion notation consist entirely of invariant quaternions, quaternion tensors, and scalars. The invariance is with respect to any arbitrary invertable, smooth, space-time transformation. This leads to considering at least four distinct kinds of quaternions, as we shall see. The one of interest for gravitation is the 4-vector invariant quaternion defined as follows:

$$V \equiv V^\mu b_\mu \tag{2.1}$$

The basis elements b_μ are defined to be linear combinations of the quaternion elements e_μ, and the vector components V^μ transform as contravariant 4-vector components under coordinate transformation from $\{x^\mu\}$ to $\{x'^\mu\}$, i.e.,

$$b_\mu \equiv b_\mu^{(\alpha)} e_\alpha \tag{2.2}$$

$$V'^\mu \equiv \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \tag{2.3}$$

In order for the quaternion V to be invariant we require that $\{b_\mu^{(\alpha)}\}$ transform as covariant 4-vector components

$$b_\mu'^{(\alpha)} \equiv \frac{\partial x^\nu}{\partial x'^\mu} b_\nu^{(\alpha)} \tag{2.4}$$

so that

$$V = V'^\mu b_\mu' = V^\nu \frac{\partial x'^\mu}{\partial x^\nu} b_\lambda^{(\alpha)} \frac{\partial x^\lambda}{\partial x'^\mu} e_\alpha = \delta_\nu{}^\lambda V^\nu b_\lambda^{(\alpha)} e_\alpha = V^\nu b_\nu \tag{2.5}$$

As a consequence, the invariant quaternion e_α can be thought of as a 4-vector quaternion since

$$e_\alpha = b^\mu_{(\alpha)} b_\mu \tag{2.6}$$

The raising (and lowering) of indices μ is done with

$$(b_\mu | b_\nu) \equiv g_{\mu\nu} \equiv \tfrac{1}{2}(b_\mu{}^+ b_\nu + b_\nu{}^+ b_\mu), \qquad g^{\mu\nu} g_{\nu\lambda} \equiv \delta_\lambda{}^\mu \equiv g_\lambda{}^\mu \tag{2.7}$$

and the raising (and lowering) of indices (α) is done with

$$\eta_{(\alpha\beta)} \equiv \tfrac{1}{2}(e_\alpha{}^+ e_\beta + e_\beta{}^+ e_\alpha) \leftrightarrow \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{2.8}$$

It follows that

$$g_{\mu\nu} = b_\mu^{(\alpha)} b_\nu^{(\beta)} \eta_{(\alpha\beta)} = b_\mu^{(\alpha)} b_{\nu(\alpha)} \tag{2.9}$$

and

$$b^\mu_{(\alpha)} = g^{\mu\nu} \eta_{(\alpha\beta)} b_\nu^{(\beta)} \tag{2.10}$$

If we use pseudocartesian coordinates, i.e., $g_{\mu\nu} \to \eta_{(\alpha\beta)}$ as the curvature approaches zero, we see that the curvature information is contained in the ten quantities $g_{\mu\nu}(x)$ or equivalently in the sixteen quantities $b_\mu^{(\alpha)}(x)$.

For flat space-time one can always find a coordinate system for which $b_\mu^{(\alpha)} = \delta_\mu^{(\alpha)}$, i.e., one in which the e_μ form the 4-vector basis. According to the Ricci identity, intrinsically curved space time is characterized by the non-commutativity of covariant differentiation on a component of any 4-vector. Specifically

$$D_\nu D_\mu V_\pi - D_\mu D_\nu V_\pi = R^\lambda_{\pi\mu\nu} V_\lambda \neq 0 \tag{2.11}$$

where $R^{\lambda}_{\pi\mu\nu}$ is the curvature tensor (Rindler, 1969). We define

$$D_\nu D_\mu V_\pi \equiv V_{\pi|\mu\nu} \frac{\partial V_\pi}{\partial x^\mu} \equiv V_{\pi,\mu} \equiv \partial_\mu V_\pi \tag{2.12}$$

and

$$D_\nu(D_\mu V_\pi) \equiv (D_\mu V_\pi)_{,\nu} - (D_\lambda V_\pi)\Gamma^{\lambda}_{\mu\nu} - (D_\mu V_\nu)\Gamma^{\lambda}_{\pi\nu} \tag{2.13}$$

From the relationships between the Christoffel symbols of the first and second kind, and $g_{\mu\nu}$ and equation (2.9), we have

$$\Gamma^{\lambda}_{\mu\nu} = g^{\pi\lambda}[\mu\nu, \pi] = g^{\lambda\pi} \tfrac{1}{2}(\partial_\mu g_{\nu\pi} + \partial_\nu g_{\pi\mu} - \partial_\pi g_{\mu\nu}) \tag{2.14}$$

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} = {} & \tfrac{1}{2} b^{\lambda(\beta)} (\partial_\mu b_{\nu(\beta)} + \partial_\mu b_{\mu(\beta)}) \\ & + \tfrac{1}{2} b^{\lambda(\beta)} b^{\pi}_{(\beta)} [b^{(\alpha)}_{\mu} (\partial_\nu b_{\pi(\alpha)} - \partial_\pi b_{\nu(\alpha)}) \\ & + b^{(\alpha)}_{\nu} (\partial_\mu b_{\pi(\alpha)} - \partial_\pi b_{\mu(\alpha)})]\end{aligned} \tag{2.15}$$

$$\begin{aligned}D_\mu b^{(\alpha)}_{\nu} = {} & \partial_\mu b^{(\alpha)}_{\nu} - \tfrac{1}{2}(\partial_\mu b^{(\alpha)}_{\nu} + \partial_\nu b^{(\alpha)}_{\mu}) \\ & + \tfrac{1}{2} b^{\pi(\alpha)} [b^{(\beta)}_{\mu} (\partial_\nu b_{\pi(\beta)} - \partial_\pi b_{\nu(\beta)}) \\ & + b^{(\beta)}_{\nu} (\partial_\mu b_{\pi(\beta)} - \partial_\pi b_{\mu(\beta)})]\end{aligned} \tag{2.16}$$

In flat space-time (accelerated frames, but no gravity) there exists a transformation back to cartesian coordinates

$$b^{(\alpha)}_{\mu} = \frac{\partial x'^{\beta}}{\partial x^\mu} \delta^{(\alpha)}_{\beta} = \frac{\partial x'^{(\alpha)}}{\partial x^\mu} = \partial_\mu x'^{(\alpha)} \tag{2.17}$$

and therefore

$$\partial_\pi b^{(\alpha)}_{\nu} = \partial_\pi \partial_\nu x'^{(\alpha)} = \partial_\nu \partial_\pi x'^{(\alpha)} = \partial_\nu b^{(\alpha)}_{\pi} \tag{2.18}$$

As a result we get

$$\Gamma^{\lambda}_{\mu\nu} = b^{\lambda(\beta)}(\partial_\mu b_{\nu(\beta)}) \qquad \text{and} \qquad D_\mu b^{(\alpha)}_{\nu} = 0 \tag{2.19}$$

We readily see the inequivalence of accelerated frames and real gravitational fields.

3. *The Gravitation Field Equation*

We are naturally led (by the fact that curvature implies $D_\mu D_\nu - D_\nu D_\mu$ operating on a 4-vector does not give zero) to a field equation of the type

$$(D_\mu D_\nu - D_\nu D_\mu) b^{(\alpha)}_{\pi} \neq 0 \tag{3.1}$$

From the quaternion invariance principle we must write this in invariant quaternion form. The simplest way to do this is

$$b^\mu (D_\mu D_\nu - D_\nu D_\mu) b^\nu = -\kappa b^\mu M_{\mu\nu} b^\nu \tag{3.2}$$

This is a natural candidate for the curvature-inducing field equation.

The equations which seem to describe physical reality are more restricted than those allowed by the invariance principle alone. We introduce a second basic postulate called the *Lorentz Principle.* We make a mathematical transformation of the invariant quaternion equation. This transformation is mathematical and not physical though historically it was developed from so-called Lorentz transformation between inertial frames in flat space-time, requiring the form invariance of allowed laws under such transformations. The generalization to arbitrary transformation covariance caused Einstein to adopt the name General Relativity, a misnomer which has stuck in the literature of the Western World. The equivalence principle notwithstanding, accelerated reference frames belong to the relativity theory and do not have any real connection with curved space time. The mathematics just look superficially similar. The 'similarity' was apparently very helpful to Einstein, but can be misleading since there are many subtle differences.

For the quaternion formalism, the Lorentz principle can be stated as follows. There are four (at least) invariant, quaternion types, distinguished by their Lorentz properties

$$\begin{aligned} &\text{4-vector } V \equiv V^\mu b_\mu \to V' \equiv L^* VL, && b_\mu = b_\mu^{(\alpha)}(x)\, e_\alpha \\ &\text{axial 4-vector } \mathscr{a} \equiv \mathscr{a}^\mu a_\mu \to \mathscr{a}' \equiv L^+ \mathscr{a} L, && a_\mu = a_\mu^{(\alpha)}(x)\, e_\alpha \\ &a\text{-spinor } \psi_a \equiv \psi_a{}^\mu k_\mu \to \psi_a{}' \equiv L^+ \psi_a, && k_\mu = k_\mu^{(\alpha)}(x)\, e_\alpha \\ &v\text{-spinor } \psi_v \equiv \psi_v{}^\mu l_\mu \to \psi_v{}' \equiv L^* \psi_v, && l_\mu = l_\mu^{(\alpha)}(x)\, e_\alpha \end{aligned} \tag{3.3}$$

here $L \equiv L^{(\alpha)} e_\alpha$, $LL^+ \equiv e_0 = L^+ L$. Only those equations which are form invariant under the above formal Lorentz transformations can describe physical laws.

For example:

$$W' V' \equiv (L^* WL)(L^* VL) = L^* WLL^* VL \tag{3.4}$$

whereas

$$W'^+ V' \equiv (L^* WL)^+ (L^* VL) = L^+ W^+ L^{+*} L^* VL = L^+(W^+ V)L \tag{3.5}$$

Therefore, to satisfy both the invariance principle and Lorentz principle, we would guess that the gravity equation takes the form

$$b^{\mu +}(D_\mu D_\nu - D_\nu D_\mu) b^\nu = -\kappa b^{\mu +} M_{\mu\nu} b^\nu \tag{3.6}$$

where κ is a coupling constant and $M_{\mu\nu}$ relates to matter in the universe. The application of the invariance and Lorentz principles together with the definition of curvature (Riemannian space) do not give equation (3.6) uniquely, but it is the simplest compatable equation.

We show now that it almost gives Einstein's gravitation equation. To do this we note

$$e_\alpha \equiv b^\mu_{(\alpha)} b_\mu = b^\mu_{(\alpha)} b_\mu^{(\beta)} e_\beta \Rightarrow b^\mu_{(\alpha)} b_\mu^{(\beta)} = \delta_\alpha{}^\beta \tag{3.7}$$

and

$$b_\mu \equiv b_\mu^{(\alpha)} e_\alpha = b_\mu^{(\alpha)} b^\nu_{(\alpha)} b_\nu \Rightarrow b_\mu^{(\alpha)} b^\nu_{(\alpha)} = \delta_\mu{}^\nu \tag{3.8}$$

Equation (3.6) can be written:

$$b^{\mu(\alpha)}(b^{\nu(\beta)}_{|\nu\mu} - b^{\nu(\beta)}_{|\mu\nu})\, e_\alpha{}^+ e_\beta = -\kappa b^{\mu(\alpha)} M_{\mu\nu} b^{\nu(\beta)} e_\alpha{}^+ e_\beta \tag{3.9}$$

Therefore, from the independence of the e_α's we have the possibility

$$b^{\mu(\alpha)}(b^{\nu}_{(\beta)|\nu\mu} - b^{\nu}_{(\beta)|\mu\nu}) = -\kappa b^{\mu(\alpha)} M_{\mu\nu} b^{\nu}_{(\beta)} \tag{3.10}$$

But by the nature of curvature

$$b^{\pi}_{(\beta)|\nu\mu} - b^{\pi}_{(\beta)|\mu\nu} = R^{\pi}_{\lambda\nu\mu} b^{\lambda}_{(\beta)} \tag{3.11}$$

hence

$$b^{\nu}_{(\beta)|\nu\mu} - b^{\nu}_{(\beta)|\mu\nu} = R^{\nu}_{\lambda\nu\mu} b^{\lambda}_{(\beta)} \equiv R_{\lambda\mu} b^{\lambda}_{(\beta)} \tag{3.12}$$

We can 'peel off' the b's in equation (3.10) by using equations (3.7) and (3.8)

$$b_{\sigma(\alpha)} b^{\mu(\alpha)} R_{\lambda\mu} b^{\lambda}_{(\beta)} b^{(\beta)}_{\pi} = -\kappa b_{\sigma(\alpha)} b^{\mu(\alpha)} M_{\mu\nu} b^{\nu}_{(\beta)} b^{(\beta)}_{\pi} \tag{3.13}$$

therefore

$$R_{\pi\sigma} = -\kappa M_{\sigma\pi} \tag{3.14}$$

We come up against the same problem Einstein faced. Though equation (3.14) is the simplest candidate, it can be shown (Bianchi identities) that

$$D^{\pi} R_{\pi\sigma} = \tfrac{1}{2} D_{\sigma} R \tag{3.15}$$

whereas, if one accepts conservation of energy, we have

$$D^{\pi} M_{\pi\sigma} = 0 \tag{3.16}$$

Equation (3.16) requires that equation (3.14) be replaced by

$$R_{\pi\sigma} - \tfrac{1}{2} g_{\pi\sigma} R = -\kappa M_{\sigma\pi} \tag{3.17}$$

We can then show that

$$R \equiv R_{\pi}{}^{\pi} = b^{\mu(\alpha)}(b^{\lambda}_{(\alpha)|\lambda\mu} - b^{\lambda}_{(\alpha)|\mu\lambda}) \tag{3.18}$$

so that equation (3.17) becomes

$$b^{(\alpha)}_{\pi}(b^{\lambda}_{(\alpha)|\lambda\mu} - b^{\lambda}_{(\alpha)|\mu\lambda}) - \tfrac{1}{2} b^{(\alpha)}_{\pi} b_{\mu(\alpha)} b^{\nu(\beta)}(b^{\lambda}_{(\beta)|\lambda\nu} - b^{\lambda}_{(\beta)|\nu\lambda}) = -\kappa M_{\pi\mu} \tag{3.19}$$

By essentially reversing the process that led to equation (3.14) we obtain from equation (3.19)

$$b^{\mu+}(D_\mu D_\nu - D_\nu D_\mu) b^{\nu} = +\kappa b^{\mu+} M_{\mu\nu} b^{\nu} \tag{3.20}$$

which differs from equation (3.6) only by the sign reversal of the right-hand side! From the weak field limit and Newton's gravity equation one obtains (e.g., Rindler, 1969),

$$\kappa = \frac{8\pi G}{c^4} \tag{3.21}$$

The Lorentz principle to be applied to equation (3.20) requires that we define the Lorentz properties of $b^{\mu+} M_{\mu\nu} b^{\nu}$

$$M \equiv b^{\mu+} M_{\mu\nu} b^{\nu} \to M' \equiv L^{+} M L \qquad \text{if} \qquad M_{\mu\nu} = M_{\mu\nu}{}^{+} \tag{3.22}$$

since

$$V_I'^{+} V_{II}' = L^{+} V_I^{+} L^{+*} L^{*} V_{II} L = L^{+}(V_I^{+} V_{II}) L = b'^{\mu+} V'_{I\mu} V'_{II\nu} b'^{\nu} \tag{3.23}$$

For a matter distribution consisting of zero pressure gas one has

$$M_{\mu\nu}(x) = \rho_0(x)\, U_\mu(x)\, U_\nu(x), \qquad U_\mu \equiv c\frac{dx_\mu}{ds}, \qquad ds \equiv [(dx|dx)]^{1/2} \quad (3.24)$$

where ρ_0 is the local dust rest frame mass density and U_μ is the 4-velocity.

We have seen that the usual tensor formulation of Einstein's gravitation equation can be converted to quaternion form and that this formulation has a very intuitive form. In the following section we generalize Einstein's work, using the quaternion formalism, and then convert the generalized gravity equation back into tensor notation, since this is more commonly used.

4. *Hypermass Generalization of the Gravitation Field Equation*

We now consider how equation (3.20) is generalized when the local rest frame mass density is an invariant quaternion

$$\rho_0 \to \rho_0^{(\alpha)}(x)\, e_\alpha, \qquad \alpha = 0, 1, 2, 3 \tag{4.1}$$

We can satisfy the invariance principle simply by writing

$$b^{\mu +} D_{\mu\nu}\, b^\nu = +\kappa b^{\mu +}\, U_\mu\, \rho_0^{(\alpha)}\, e_\alpha\, U_\nu\, b^\nu \tag{4.2}$$

but since $L^{+*}\rho_0^{(\alpha)} e_\alpha L^* \neq e_0$ we do not satisfy the Lorentz principle. If we place ρ_0 to the right or left of $b^{\mu +}\, U_\mu\, U_\nu\, b^\nu$ the same problem is encountered, L and e_k do not commute. We seem to have no other choice than to require $\kappa\rho_0$ to be proportional to e_0 so that it commutes with L. This in turn requires that the coupling constant κ be an invariant quaternion, $\kappa = \kappa^{(\beta)} e_\beta$. The simplest generalization of the gravity equation is then ($D_{\mu\nu} \equiv D_\mu D_\nu - D_\nu D_\mu$)

$$\begin{aligned} b^{\mu +} D_{\mu\nu}\, b^\nu &= b^{\mu +}\, U_\mu\, \kappa^{(\alpha)}\, \rho_{0(\alpha)}\, U_\nu\, b^\nu \\ &= \kappa^0\, b^{\mu +}\, U_\mu\, \rho_{0(0)}\, U_\nu\, b^\nu + \kappa^{(k)}\, b^{\mu +}\, U_\mu\, \rho_{0(k)}\, U_\nu\, b^\nu \end{aligned} \tag{4.3}$$

In tensor notation this gives

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\kappa^{(0)} M_{\mu\nu(0)} - \kappa^{(k)} M_{\mu\nu(k)} \tag{4.4}$$

where $M_{\mu\nu(\alpha)} \equiv U_\mu \rho_{0(\alpha)} U_\nu$. We get back the usual expression except for an added source term $\kappa^{(k)} M_{\mu\nu(k)}$. For pressureless gas, we see that the four divergence of the right-hand side is still zero as required. With reference to the usual cosmological constant we note (suggested by Len Rosen) that if

$$\Lambda g_{\mu\nu} \equiv \kappa^{(k)} M_{\mu\nu(k)} \tag{4.5}$$

then

$$\Lambda = \tfrac{1}{4}\kappa^{(k)}\, \rho_{0(k)}\, c^2 \tag{4.6}$$

5. *Conclusion*

The two general postulates we have discussed, the invariance principle and the Lorentz principle, have shown that Einstein's gravity equation is a simple and natural choice for space-time curvature. These principles also guided us to a generalization of Einstein's equation for hypermass which required generalization of the gravitational coupling constant, G, to a quaternion. The resulting minor modification of the gravity equation shows that hypermass is unlikely to have any observable cosmological consequences.

The formalism developed, however, provides the ground work for developing relativistic quantum equations in curved space-time and rather arbitrary curvilinear coordinate systems. This is very important, considering the nonlinearity of gravitation and the central importance of the quantum superposition principle. The ultimate foundations of quantum theory have to be compatible with a curved space-time medium even if curvature effects have a negligible effect on practical calculations and microscopic interactions. The validity of the superposition principle is also important in the continuing investigations of CP and CPT conservation.

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